International Journal of Theoretical Physics, Vol. 32, No. 3, 1993

A Model of Piron's Preparation–Question Structures in Ludwig's Selection Structures

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Received July 21, 1992

We give a model of the basic Jauch–Piron (JP) approach to quantum physics, i.e., of "preparation–question structure" (with four basic axioms and without axioms C, P, A), in terms of Ludwig's "selection structure"; in the latter structure the primitive notion of "individual sample" of a physical entity is formally described (without making reference to any probability concept). Once we interpret Piron's concept of "question" in Ludwig's context of a selection structure, we find that there is no difficulty in formalizing notions such as "performable together questions"; moreover, results such as "$\alpha^{\sim\sim}=\alpha$" or "$(\alpha\triangle\beta)^{\sim}=\alpha^{\sim}\nabla\beta^{\sim}$" can be formally proved. We develop the theory along the lines of the JP approach; the set of JP propositions is derived and it turns out to be a complete lattice, as happens in Piron's theory, but with a different physical interpretation of the lattice operations. Finally, we study some connections between the standard Ludwig foundation and our approach.

1. PRELIMINARY REMARKS

There are many axiomatizations of quantum mechanics. We cite the foundational work of Mackey (1963), the operational approach of Davies and Lewis (1970; Davies, 1976), the algebraic approach to quantum field theory of Haag and Kastler (1964), the Jauch and Piron (1969; Piron, 1972, 1976*a*,*b*, 1981) foundation of quantum physics, and the axiomatic foundation of Ludwig (1977, 1983; Neumann, 1983). A feature characterizing almost all of these approaches is the use of probability as a basic concept of the theory. The Jauch–Piron (JP) work (1969) can be distinguished among these approaches because it has been successful, especially as presented in Piron (1976*b*), in recovering the full quantum formalism "without making use of the notion of probability." This approach to quantum physics is based on the following concept of *question* (Piron, 1972):

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0020-7748/93/0300-0407$07.00/0

(DQ) "We define a question to be a measurement (or experiment) leading to an alternative of which the terms are 'yes' and 'no'" when it is performed on a *single sample* of the physical entity.

The idea of *occurrence of a question* is implicit in the theory:

(OQ) If the performing of a question α gives the answer "yes" (respectively, "no") for an individual sample, we shall say that α *occurs* (respectively, *does not occur*).

This idea is very useful in order to explain the physical meaning of the axioms of the original JP theory. The real core of the JP theory makes use of the following dyadic predicate *true* involving *preparations* of individual samples (under well-defined and repeatable conditions) and *questions* (Piron, 1976*a*):

(TQ) "When the physical system has been prepared in such a way that the physicist may affirm that in the event of an experiment the result 'yes' is certain, we shall say that the question is 'true'."

All this shows that in the JP approach we have two distinct levels of description: the first one (DQ), which pertains to *individual samples*, leads, for a question, to the alternatives "yes" or "no" (*single test* of the question on an individual sample); the second level (TQ), which pertains to a *preparation* as a whole, assigns to a question the value "true" if, for any individual sample so prepared, the answer "yes" to the test of the question is certain (*elementary experiment* of a pair preparation–question).

In Cattaneo *et al.* (1988, 1989) and Cattaneo and Nisticò (1991) we presented a formalization of the JP approach based on two *classes* of abstract signs and a *two-argument predicate* sign. The two classes of signs are physically interpreted as describing the *primitive undefined notions* of "preparation" of individual samples (formally denoted by $x, y, \ldots$) and of "question" (formally denoted by $\alpha, \beta, \ldots$), respectively; the two-argument predicate sign is physically interpreted as "question *true* in a preparation" [formally denoted by $(x, \alpha)T$].

The notion (OQ) of the occurrence of a question cannot be translated into the mathematical language of our formalized theory since no formal object describing an individual sample is introduced, neither as a primitive notion or as a derived concept. To be precise, no formula of the kind $(i, \alpha)Y$ [respectively, $(i, \alpha)N$], semantically interpreted as "the individual sample i produces the answer 'yes' (respectively, 'no') in a test of the question α," is allowable in our formalization of JP theory.

This lack in the formalization has several negative consequences, the most important being, in our opinion, the following three.

(a) In general, any statement proved in the JP framework by making use of (OQ) is no longer a *theorem* in our formalized JP theory; indeed, it may happen that for a statement of this kind a formal proof (i.e., a proof which uses only formalized concepts) does not exist. In this case, the only way of formalizing the statement is that of introducing it as an axiom, and this can be done when (OQ) does not explicitly appear.

Example 1.1. Piron once presented the following *definition*:

Definition 1.2. "If α is a question, we denote by $\alpha^{\sim}$ the question obtained by exchanging the terms of the alternative" (Piron, 1976*b*).

Jauch and Piron make the following statement:

Proposition 1.3. "It is clear that $\alpha = (\alpha^{\sim})^{\sim}$" (Jauch and Piron, 1969).

We see that a proof of Proposition 1.3 cannot be given without making use of the notion of the occurrence of a question; therefore, although physically evident, we cannot find a mathematical proof in the formalized JP theory which does not contain (OQ). May we establish Proposition 1.3 as an axiom? No, because the definition of the question $\alpha^{\sim}$ requires the notion of the occurrence of a question, too.

(b) Another kind of problem arises when we face the description of two or more *performable together questions.* Let us quote Aerts (1982):

> We shall say that we can perform both questions α and β together iff there exists an experiment $E(\alpha, \beta)$ having four outcomes that we shall label by {yes,yes}, {yes,no}, {no,yes} and {no,no}, [. . .]. We will define the question
>
> $\alpha \triangle \beta$ which consists of performing the experiment E and attributing the answer "yes" if we get the outcome {yes,yes}. [. . .]
>
> $\alpha \triangledown \beta$ which consists of performing the experiment E and attributing the answer "yes" if we get the outcomes {yes,yes}, {yes,no}, or {no,yes}. [. . .]
>
> $\alpha \ominus \beta$ which consists of performing the experiment E and attributing the answer "yes" if we get the outcomes {yes,yes} or {no,no}. [. . .]

This definition of performable together questions and of new questions $\alpha \triangle \beta$, $\alpha \triangledown \beta$, $\alpha \ominus \beta$ is physically correct. However, it is also true that it cannot be translated into the mathematical language of our JP formalized theory.

(c) Another fundamental shortcoming of our formalized JP theory, and of customary quantum mechanics as well, is that "an individual experimental result, e.g., an individual trace in a cloud chamber, cannot be compared with the theory [. . .]. For instance, it is not possible to translate into mathematical language such propositions as 'the position of this individual electron has been measured in the region $\mathscr{V}$'" (Ludwig, 1977).

On the contrary, Ludwig's (1977, 1983) foundations of quantum mechanics has the merit that "every experimental result (also if concerning an individual microsystem) can be described by a mathematical relation." Precisely, Ludwig's approach is *mathematically* based on a "selection structure" on a set M; the points of M are *physically interpreted* as "images" of individual samples of a physical entity (in this way, semantically, the notion of "individual sample" is a primitive one). "This general framework allows us to consider single physical systems, and not only ensembles, in the mathematical picture" (Neumann, 1983). We stress that at the level of selection structure no use of statistical concepts is made.

Obviously, the whole Ludwig theory has a stronger mathematical structure (involving in particular a mapping λ with a physical interpretation of *relative probability*) which characterizes it as a *statistical theory*. Quoting Neumann (1983), "Probability is introduced as a relative (or conditional) probability of two experimentally well-defined procedures employed for the selection of the objects under consideration. The mathematical scheme so developed is called a structure of 'statistical selection procedures'."

In the present work we only take into account Ludwig's "selection structure." Our aim is to make a first step along the direction of formalizing Piron's approach starting from the primitive notion of "individual sample" of a physical entity. We believe that Ludwig's "selection structure" is the suitable framework to reach this result. In particular, we produce an *interpretation* of Piron's "preparation–question structure" in Ludwig's "selection structure" in such a way that the four basic axioms of Piron's approach, under the interpretation, turn out to be true (i.e., theorems) in Ludwig's structure. In other words, we construct a *model* of the basic Piron theory (without Axioms C, P, A) in Ludwig's selection structure (without statistical notions).

2. LUDWIG'S SELECTION STRUCTURES AND JP QUESTIONS

In this section we introduce the Ludwig's *selection structure* as the more suitable environment to assent to the "*program of realism*" invoked by JP and consisting in giving "a complete description of each individual system as it is in all its complexity" (Piron, 1976*b*). According to Ludwig (1977), "To give a **mathematical picture** of such experiments on *individual systems*, it is necessary to introduce a set M, the elements of which shall be 'images' of the system"; the **physical interpretation** of the formula "$i \in M$" is as follows:

> "Given a special atom i in an experiment, the relation $i \in M$ should be the mathematical form of the proposition: i is a physical system.

However, the relation $i \in M$ reflects the proposition 'i is a physical system' only if the set M is endowed with a structure as an image of *preparing* and *recording procedures*" (Ludwig, 1977).

Definition 2.1. A structure of selection procedures, or briefly a *selection structure*, is a pair $\langle M, \mathcal{S} \rangle$, where M is a nonempty set and $\mathcal{S}$ is a subset of the power set $\mathcal{P}(M)$ such that the following conditions hold ($x \backslash y$ being the relative complement of x in y):

(S1) $x, y \in \mathcal{S}$ and $y \subseteq x$ imply $x \backslash y \in \mathcal{S}$.
(S2) $x, y \in \mathcal{S}$ implies $x \cap y \in \mathcal{S}$.

A *complete selection structure* is a selection structure in which condition (S2) is substituted by the following condition:

(S2c) $\{x_j\} \subseteq \mathcal{S}$ implies $\bigcap x_j \in \mathcal{S}$.

According to Ludwig (1977), "The *physical interpretation* of '$i \in x$ and $x \in \mathcal{S}$' is as follows:

> The physical system i has been selected by the selection procedure x. In this sense, an element $x \in \mathcal{S}$ represents the method of selecting as well as the family of single samples of the physical system selected by this method."

Of course, the empty set $\varnothing$ is an element of $\mathcal{S}$, called the *trivial* selection procedure. We set $\mathcal{S}' := \mathcal{S} \backslash \{\varnothing\}$ in the sequel. If $x \in \mathcal{S}$, then the set $\mathcal{S}(x) := \{y : y \in \mathcal{S}$ and $y \subseteq x\}$ is a Boolean algebra of sets, called the *laboratory* induced by x.

2.1. Questions and Selection–Question Structures

The selection structure $\langle M, \mathcal{S} \rangle$ is the basic mathematical structure on which Ludwig has founded his approach to QP. Now, we introduce in this Ludwig framework the **derived concept** of a *question*, according to Piron's approach, giving in this way a link between these two approaches (from now on, for the sake of simplicity, we denote selection structures by $\mathcal{S}$):

Definition 2.2. Let $\mathcal{S}$ be a complete selection structure; a *question* for $\mathcal{S}$ is any mapping defined in $\mathcal{S}$, $\alpha : \mathcal{D}_\alpha \mapsto \mathcal{S}$, satisfying the following conditions.

(i) $\mathcal{D}_\alpha$, the definition domain of the question α, is a subset of $\mathcal{S}$ such that:

(a) $\varnothing \in \mathcal{D}_\alpha$.
(b) $\{x_i\} \subseteq \mathcal{D}_\alpha$ implies $\bigcap_i x_i \in \mathcal{D}_\alpha$.
(ii) For every $x \in \mathcal{D}_\alpha$, $\alpha(x) \subseteq x$.
(iii) For every $x_1, x_2 \in \mathcal{D}_\alpha$, $\alpha(x_1 \cap x_2) = \alpha(x_1) \bigcap x_2$.

If α is a question, we say that α is *performable* in the preparation x iff $x\in\mathscr{D}'_\alpha$; moreover, for every $x\in\mathscr{D}'_\alpha$ the pair (x, α) is called the yes–no *elementary experiment* consisting of the *preparing process* x and of the *question* α, with associated *recording* (or *counting*) *process* $\alpha(x)$.

For every yes–no elementary experiment (x, α), we define the *occurrence function* as the characteristic functional of the subset $\alpha(x)$, restricted to x, $t_{(x,\alpha)}: x \mapsto \{0, 1\}$, defined by the law

$$\forall i\in x, \quad t_{(x,\alpha)}(i) := \begin{cases} 1 & \text{if } \ i\in\alpha(x) \\ 0 & \text{if } \ i\in x\backslash\alpha(x) \end{cases}$$

The *physical interpretation* of the yes–no experiment "(x, α)" is as follows:

> x is a procedure to prepare individual samples of the physical entity and α is a question which is performable on x and which tests any single sample $i\in x$ giving the answer "yes" if $t_{(x,\alpha)}(i)=1$ and "no" if $t_{(x,\alpha)}(i)=0$.

The *physical interpretation* of the formula "$i\in\alpha(x)\subseteq x$" (respectively, "$j\in x\backslash\alpha(x)\subseteq x$") is as follows:

> i (respectively, j) is an individual sample from the preparation x which has produced the answer "yes" (respectively, "no") in a single test of the question α.

In this way, the performance of an "elementary experiment" (x, α) consists in the "preparation" of an ensemble x of individual samples $i\in x$ of the physical entity and in the "test" on each of these samples of the question α, producing the subensemble $\alpha(x)$ [respectively, $x\backslash\alpha(x)$] of those samples which have given the answer "yes" (respectively, "no"). The dichotomic "yes–no" behavior of the question α in a single test is formalized by the trivial properties $\alpha(x)\cap(x\backslash\alpha(x))=\varnothing$ and $\alpha(x)\cup(x\backslash\alpha(x))=x$. Note that if $x\in\mathscr{D}_\alpha$, then in general $\alpha(x)$ is not an element of $\mathscr{D}_\alpha$, corresponding to the fact that the execution of the question α on a sample can destroy the sample under discussion.

On the basis of the above interpretations, the meaning of conditions (i) and (ii) in Definition 2.2 turns out to be clear. Now, it is easy to prove that condition (iii) is equivalent to the following:

> (iii)′ Let $x_1, x_2\in\mathscr{D}_\alpha$; then for every $i\in x_1\cap x_2$, $t_{(x_1,\alpha)}(i)=t_{(x_2,\alpha)}(i)$.

Thus, the interpretation of condition (iii) in Definition 2.2 is simply that the outcome of a test of the question α is fixed for every individual sample $i\in M_\alpha=\bigcup_{x\in\mathscr{D}_\alpha} x$ and it is independent of the particular preparation x (with $i\in x$). Therefore, our notion of question agrees with the (DQ) statement. Moreover, the notion (OQ) of "occurrence of a question" in a single test finds an appropriate formalization: $\forall\alpha$, $\forall i \in x \subseteq M_\alpha$, $(i, \alpha)Y$ [respectively, $(i, \alpha)N$] iff $t_{(x,\alpha)}(i) = 1$ (respectively, $=0$).

The *trivial question*, denoted by $\mathbf{O}$: $\{\varnothing\} \mapsto \mathscr{S}$, is defined by $\mathbf{O}(\varnothing) = \varnothing$ (hence a question α is the trivial question iff $\mathscr{D}_\alpha = \{\varnothing\}$). We call *null* question any question α such that $\forall x \in \mathscr{D}_\alpha$, $\alpha(x) = \varnothing$. For the sake of brevity we put in the sequel $M_\alpha := \bigcup_{x \in \mathscr{D}_\alpha} x \subseteq M$ and $\mathscr{D}'_\alpha := \mathscr{D}_\alpha \setminus \{\varnothing\}$.

Remark 2.3. Trivially, any question is a monotonic mapping [i.e., for every x_1, $x \in \mathscr{D}_\alpha$, $x_1 \subseteq x$ implies $\alpha(x_1) \subseteq \alpha(x)$] such that for every family $\{x_i\} \subseteq \mathscr{D}_\alpha$, $\alpha(\bigcap_i x_i) = \bigcap_i \alpha(x_i)$.

In the above definition, under conditions (i) and (ii), condition (iii) is equivalent to the following two conditions:

(iiia) The mapping $\alpha^\sim$: $\mathscr{D}_\alpha \mapsto \mathscr{S}$, $x \to \alpha^\sim(x) := x \setminus \alpha(x)$ is a question, called the *inverse* question of α.

(iiib) For every x_1, $x_2 \in \mathscr{D}_\alpha$, $\alpha(x_1 \cap x_2) = \alpha(x_1) \cap \alpha(x_2)$ and $\alpha^\sim(x_1 \cap x_2) = \alpha^\sim(x_1) \cap \alpha^\sim(x_2)$.

Note that according to the proposed formalization, the statement "$\alpha^{\sim\sim} = \alpha$" is a theorem.

Definition 2.4. A *preparing-question structure* (pq-s) is a structure $\mathbf{PQ} := \langle \mathscr{S}, \mathscr{Q}; \mathbf{O}; I, {}^\sim \rangle$ where:

- (AX1) $\mathscr{S}$, the set of *preparations* of $\mathbf{PQ}$, is a complete selection structure.
- (AX2) $\mathscr{Q}$, the set of *questions* of $\mathbf{PQ}$, is a set of questions $\alpha: \mathscr{D}_\alpha \mapsto \mathscr{S}$ (it is not required that $\mathscr{Q}$ contains all the possible questions for $\mathscr{S}$) such that:
  - (i) The trivial question $\mathbf{O}$ belongs to $\mathscr{Q}$.
  - (ii) ${}^\sim$ and I are unary operations on $\mathscr{Q}$ which to every question $\alpha \in \mathscr{Q}$ associate the inverse question $\alpha^\sim$ and the relative certain question I_α of α respectively defined by:
    - (a) $\alpha^\sim: \mathscr{D}_\alpha \mapsto \mathscr{S}$, where $\forall x \in \mathscr{D}_\alpha$, $\alpha^\sim(x) := x \setminus \alpha(x)$;
    - (b) $I_\alpha: \mathscr{D}_\alpha \mapsto \mathscr{S}$, where $\forall x \in \mathscr{D}_\alpha$, $I_\alpha(x) := x$.

    (i.e., for any question $\alpha \in \mathscr{Q}$, both questions $\alpha^\sim$, $I_\alpha \in \mathscr{Q}$).

A pq-s is said to be surjective iff the further conditions holds:

(AX3) $\mathscr{S} = \bigcup_{\alpha \in \mathscr{Q}} \mathscr{D}_\alpha$.

2.2. True Questions

According to Aerts (1982), it is now possible to formalize correctly the further statement:

Definition 2.5. We say that in the preparation $x \in \mathscr{D}_\alpha$ of the question α there is a *certain chance* to obtain the answer "yes" (respectively, "no") for α [or also, "yes" (respectively, "no") is a possible answer for α] iff there exists $i \in x$ such that $t_{(x,\alpha)}(i) = 1$ [respectively, $t_{(x,\alpha)}(i) = 0$], i.e., iff $\alpha(x) \neq \varnothing$ [respectively, $x \backslash \alpha(x) \neq \varnothing$]. Moreover, we say that in the preparation $x \in \mathscr{D}_\alpha$ the chance to have the answer "yes" (respectively, "no") for the question α is the certainty iff $\alpha(x) = x$ [respectively, $\alpha(x) = \varnothing$].

Now we can introduce the following new **derived signs** of our mathematical theory, linking preparations and questions.

Definition 2.6. We denote by T and F the signs respectively defined by

$$\begin{aligned}(x, \alpha)T \quad & \text{iff} \quad x \in \mathscr{D}'_\alpha \text{ and } \alpha(x) = x \\ & \text{iff} \quad \text{for every } i \in x,\ t_{x,\alpha}(i) = 1 \\ (x, \alpha)F \quad & \text{iff} \quad x \in \mathscr{D}'_\alpha \text{ and } \alpha(x) = \varnothing \ [\text{i.e., } \alpha^{\sim}(x) = x] \\ & \text{iff} \quad \text{for every } i \in x,\ t_{x,\alpha}(i) = 0\end{aligned}$$

Moreover, let $x \in \mathscr{D}'_\alpha$; then we set $(x, \alpha)U$ iff neither $(x, \alpha)T$ nor $(x, \alpha)F$.

For any question α we define the following three subsets of $\mathscr{S}$:

(i) $\mathscr{S}_T(\alpha) = \{x \in D_\alpha \mid (x, \alpha)T\}$.
(ii) $\mathscr{S}_F(\alpha) = \{x \in D_\alpha \mid (x, \alpha)F\}$.
(iii) $\mathscr{S}_U(\alpha) = \{x \in D_\alpha \mid (x, \alpha)U\}$.

The *physical interpretation* of formula "$(x, \alpha)T$" [respectively, "$(x, \alpha)F$"] is as follows:

> In the preparation x the question α is "true" (respectively, "false"), i.e., the chance to obtain the answer "yes" (respectively, "no") in the preparation x for the question α is the certainty.

This interpretation agrees with the (PT) Piron definition.

> "When the physical system has been prepared in such a way [x] that the physicist may affirm that in the event of an experiment [(x, α)] the result 'yes' [respectively, 'no'] is certain, we shall say that the question [α] is "true" [respectively, 'false']" (Piron, 1976*b*).

2.3. Performable Together Questions

If α and β are two questions, then in general it makes no sense to perform both questions together. Indeed, it could happen that we can execute α on the single sample $i \in x \in \mathscr{D}_\alpha$ but that we cannot execute β on the same sample since the execution of α has destroyed the sample under discussion. The possibility of performing together two questions on the samples of a certain preparation is formalized in the following definition.

Definition 2.7. Let α, β be two questions. Then we will say that:

(i) α and β are *performable together* iff $\mathscr{D}_\alpha \cap \mathscr{D}_\beta \neq \{\emptyset\}$; in this case we write $\alpha(pt)\beta$.

(ii) α and β are *globally performable together* iff $\mathscr{D}_\alpha = \mathscr{D}_\beta \neq \emptyset$; in this case we write $\alpha(gpt)\beta$.

(iii) α and β are *not performable together* iff $\mathscr{D}_\alpha \cap \mathscr{D}_\beta = \{\emptyset\}$.

So, we have two binary relations defined on the set of all questions: the relation (pt) of being performable together, which is symmetric and reflexive but in general not transitive, and the equivalence relation (gpt). For fixed $x \in \mathscr{S}$, $x \neq \emptyset$, we have the following equivalence relation on $\mathscr{Q}$.

$$\alpha(pt)_x\beta \quad \text{iff} \quad x \in \mathscr{D}_\alpha \cap \mathscr{D}_\beta$$

$$\text{iff} \quad \alpha \text{ and } \beta \text{ are performable together in } x$$

Let us consider two questions α and β which are performable together, i.e., $\mathscr{D}_\alpha \cap \mathscr{D}_\beta \neq \{\emptyset\}$, and set $M_{\alpha,\beta} := \bigcup_{x \in \mathscr{D}_\alpha \cap \mathscr{D}_\beta} x \subseteq M$, which, for the above condition, is not empty.

Definition 2.8. For any pair of performable together questions α and β, the *experiment induced* by α and β is the mapping $E_{\alpha,\beta}: M_{\alpha,\beta} \mapsto \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ defined as follows:

$$\forall i \in M_{\alpha,\beta}, \quad E_{\alpha,\beta}(i) := (t_{(x,\alpha)}(i), t_{(x,\beta)}(i))$$

For every $i \in M_{\alpha,\beta}$, the pair $(t_{(x,\alpha)}(i), t_{(x,\beta)}(i))$ is called the *outcome* of the experiment $E_{\alpha,\beta}$ produced by i.

The *physical interpretation* of experiment $E_{\alpha,\beta}$ is as follows:

Given a preparation $x \in \mathscr{D}_\alpha \cap \mathscr{D}_\beta$, both questions α and β are tested on every sample i prepared by x and the pair $(t_{x,\alpha}(i), t_{x,\beta}(i))$ is taken as the outcome of the experiment $E_{\alpha,\beta}$.

Because of the physical interpretation of the experiment $E_{\alpha,\beta}$, one expects that the relations defined in Definition 2.6. are related with the outcomes of experiments $E_{\alpha,\beta}$. Quoting Aerts (1981),

> α is true iff we are certain to get one of the outcomes {*yes,yes*} or {*yes,no*} for the experiment E.
> $\alpha^{\sim}$ is true iff we are certain to get one of the outcomes {*no,yes*} or {*no,no*} for the experiment E.
> β is true iff we are certain to get one of the outcomes {*yes,yes*} or {*no,yes*} for E and
> $\beta^{\sim}$ is true iff we are certain to get {*yes,no*} or {*no,no*} for E.

The following results show that these statements hold in our framework.

Proposition 2.9. Let α and β be two performable together questions such that $x \in \mathscr{D}_\alpha \cap \mathscr{D}_\beta$. Then the following statements hold:

(i) $(x, \alpha)T$ iff $E_{\alpha,\beta}(x) \subseteq \{(1, 1), (1, 0)\}$.
(ii) $(x, \alpha^{\sim})T$ iff $E_{\alpha,\beta}(x) \subseteq \{(0, 1), (0, 0)\}$.
(iii) $(x, \beta)T$ iff $E_{\alpha,\beta}(x) \subseteq \{(1, 1), (0, 1)\}$.
(iv) $(x, \beta^{\sim})T$ iff $E_{\alpha,\beta}(x) \subseteq \{(1, 0), (0, 0)\}$.

Proof. Indeed, $E_{\alpha,\beta}(x) \subseteq \{(1, 1), (1, 0)\}$ iff $\forall i \in x$, $E_{\alpha,\beta}(i) = (1, 1)$ or $E_{\alpha,\beta}(i) = (1,0)$ iff $\forall i \in x$, $t_{x,\alpha}(i) = 1$ iff $(x, \alpha)T$. The other statements have similar proofs. ■

2.4. Preparing-Question Structure in Ludwig's Selection Structure

For every pair, α, β of performable together questions three nontrivial mappings $\alpha \triangle \beta$, $\alpha \triangledown \beta$, and $\alpha \ominus \beta$ with domains $\mathscr{D}_{\alpha \triangle \beta} = \mathscr{D}_{\alpha \triangledown \beta} = \mathscr{D}_{\alpha \ominus \beta} = \mathscr{D}_\alpha \cap \mathscr{D}_\beta$ can be defined as follows:

$$(\alpha \triangle \beta)(x) := \alpha(x) \cap \beta(x) \tag{2.1}$$

$$(\alpha \triangledown \beta)(x) := x \backslash [\alpha^{\sim}(x) \cap \beta^{\sim}(x)] = \alpha(x) \cup \beta(x) \tag{2.2}$$

$$(\alpha \ominus \beta)(x) := [\alpha(x) \cap \beta(x)] \cup [\alpha^{\sim}(x) \cap \beta^{\sim}(x)] \tag{2.3}$$

Proposition 2.10. The following statements hold:

(i) If α and β are questions, then $\alpha(pt)_x\beta$ implies $\alpha^{\sim}(pt)_x\beta$ and $\alpha(pt)_x\alpha \triangle \beta$, $\alpha(pt)_x\alpha \triangledown \beta$, $\alpha(pt)_x\alpha \ominus \beta$.

(ii) If α, β, and γ are three questions pairwise performable together, then $\alpha(pt)_x\beta \triangle \gamma$, $\alpha(pt)_x\beta \triangledown \gamma$, and $\alpha(pt)_x\beta \ominus \gamma$, for every $x \in \mathscr{D}_\alpha \cap \mathscr{D}_\beta \cap \mathscr{D}_\gamma$.

From the physical point of view, we may assume that $\alpha \triangle \beta$, $\alpha \triangledown \beta$, and $\alpha \ominus \beta$ are new questions, i.e., concretely realizable questions starting from two performable together questions α and β. We put this in a formal way by introducing the concept of preparing-question structure in the context of Ludwig's (L) selection structure.

Definition 2.11. An L preparing-question structure (L-pq-s) is any structure $\mathbf{PQ_L} = \langle \mathcal{S}, \mathcal{Q}; \mathbf{O}; I, ^{\sim}, \triangle \rangle$, where:

(AXL2) $\mathbf{PQ} = \langle \mathcal{S}, \mathcal{Q}; \mathbf{O}; I, ^{\sim} \rangle$ is a preparing-question structure.

(AXL2) $\triangle$ is a mapping $\{\alpha_j\} \in \mathcal{P}(\mathcal{Q}) \mapsto \triangle \alpha_j \in \mathcal{Q}$, where $\mathcal{P}(\mathcal{Q})$ is the power set of $\mathcal{Q}$, such that $\mathcal{D}_{\triangle \alpha j} = \bigcap \mathcal{D}_{\alpha_j}$ and for every $x \in \mathcal{D}_{\triangle \alpha j}$, $(\triangle \alpha_j)(x) := \bigcap \alpha_j(x)$.

Proposition 2.12. Let $\langle \mathcal{S}, \mathcal{Q}; \mathbf{O}; I, ^{\sim}, \triangle \rangle$ be an L-pq-s. Then:

(i) For every family $\{\alpha_j\}$ of questions from $\mathcal{Q}$ the question $\nabla \alpha_j := (\triangle \alpha_j^{\sim})^{\sim} \in \mathcal{Q}$ exists such that

$$\mathcal{D}_{\nabla \alpha_j} = \bigcap \mathcal{D}_{\alpha_j} \qquad \text{and} \qquad \nabla \alpha_j(x) = \bigcup \alpha_j(x)$$

(ii) For every family of questions $\{\alpha_i\}$, the question $\ominus \alpha_i := (\triangle \alpha_i) \nabla (\triangle \alpha_i^{\sim}) \in \mathcal{Q}$ exists such that

$$\mathcal{D}_{\ominus \alpha_i} = \bigcap \mathcal{D}_{\alpha_i} \qquad \text{and} \qquad (\ominus \alpha_i)(x) = [(\bigcap \alpha_i)(x)] \cup [\bigcap \alpha_i^{\sim}(x)]$$

The following proposition about performable together questions, some of which appeared in Aerts (1982), can be easily formally proved in a L-pq-s.

Proposition 2.13. Let α and β be two questions that can be performed together; then for every $x \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta$

$$\begin{aligned} (x, \alpha \triangle \beta)T \quad & \text{iff} \quad (x, \alpha)T \ \text{ and } \ (x, \beta)T \\ & \text{iff} \quad \forall i \in x, \quad E_{\alpha,\beta}(i) = (1, 1) \end{aligned} \tag{2.4}$$

Moreover, $(x, \alpha \nabla \beta)T$ iff $\forall i \in x$, $E_{\alpha,\beta}(i) \in \{(0, 1), (1, 0), (1, 1)\}$, from which we get

$$(x, \alpha)T \ \text{ or } \ (x, \beta)T \ \text{ implies } \ (x, \alpha \nabla \beta)T \tag{2.5}$$

Proposition 2.14. In any L-pq-s the following statements hold:

(i) $\forall x \in \mathcal{D}_\alpha$, $\forall i \in x$, $t_{(x, I_\alpha)}(i) = 1$.

(ii) $\forall x \in \mathcal{D}_\alpha$, $\forall i \in x$, $t_{(x, \alpha^{\sim})}(i) = 1 - t_{(x, \alpha)}(i)$.

(iii) $\forall x \in \bigcap \mathcal{D}_{\alpha_j}$, $\forall i \in x$, $\forall j$, $t_{(x, \alpha_j)}(i) = 1$ iff $t_{(x, \triangle \alpha_j)}(i) = 1$ and $\exists j$, $t_{(x, \alpha_j)}(i) = 0$ implies $t_{(x, \triangle \alpha_j)}(i) = 0$.

According to the physical interpretations stated in Section 2, we have the following interpretations pertaining to the JP approach to QP (Cattaneo *et al.*, 1988, 1989; Cattaneo and Nisticò, 1992):

(PD1) For every question α the *relative certain* question I_α exists, which consists in measuring α and applying to each answer (1 or 0) of α the trivial Boolean function $1_B: \{0, 1\} \mapsto \{0, 1\}$, associating to any $i \in \{0, 1\}$ the number $1_B(i) = 1$.

(PD2) If α is a question, $\alpha^{\sim}$ is the question, called the *opposite* or the *inverse* of α, obtained by exchanging the terms of the alternative, i.e., it consists in measuring α and applying the Boolean negation to each answer of α.

(PD3) If $\{\alpha_j\}$ is a family of questions, then $\triangle \alpha_j$ is the question defined in the following manner:
(1) For every individual sample from $x \in \mathscr{D}_{\alpha_j}$ one performs every experiment corresponding to α_j.
(2) One attributes to $\triangle \alpha_j$ the answer "no" if at least one of the samples has produced the answer "no," and the answer "yes" otherwise.

Moreover, the following statements, which are hidden axioms in the JP theory (Cattaneo *et al.*, 1988), turn out to be theorems:

Proposition 2.15. In any L-pq-s the following are satisfied:

(AXJP1) $\forall x \in \mathscr{D}_\alpha$, $(x, I_\alpha)T$.

(AXJP2) $\forall x \in \mathscr{D}_\alpha$, $(x, \alpha^{\sim})F$ iff $(x, \alpha)T$.

(AXJP3) $\forall x \in \mathscr{D}_{\triangle \alpha_j}$, $(x, \triangle \alpha_j)T$ iff $\forall \alpha_j$, $(x, \alpha_j)T$ and $(x, \alpha_j)F$ implies $(x, \triangle \alpha_j)F$.

(AXJP4) There is no $x \in \mathscr{D}_\alpha$ such that $(x, \alpha)T$ and $(x, \alpha)F$.

3. JP PROPOSITIONS AND STATES

We now have the following **derived definitions** involving an L-pq-s according to Piron's approach to quantum physics.

Definition 3.1. Let $\alpha, \beta \in \mathscr{Q}$. Then we will say that:

(DF1) For any question α, $\mathscr{Q}$ contains also the inverse question $I_\alpha^{\sim}$ which is the *relative absurd question*, denoted by O_α.

(DF2) α is *less* than β, symbolized by $\alpha \prec \beta$, iff $(x, \alpha)T$ implies $(x, \beta)T$, iff $\mathscr{S}_T(\alpha) \subseteq \mathscr{S}_T(\beta)$.

(DF3) α is JP-*equivalent* to β, denoting this by $\alpha \equiv_{jp} \beta$, iff $\alpha \prec \beta$ and $\beta \prec \alpha$, iff $\mathscr{S}_T(\alpha) = \mathscr{S}_T(\beta)$; the relation $\equiv_{jp}$ is an equivalence relation on $\mathscr{Q}$.

(DF4) Any JP-equivalence class is called a JP-*proposition*.

(DF5) The quotient set $\mathscr{Q}\backslash_{\equiv_{jp}}$ of all JP propositions will be denoted by $\mathscr{L}$. The elements of $\mathscr{L}$ will be denoted by $a, b, c, \ldots$ (with indices, if necessary).

(DF6) A JP-proposition $a\in\mathscr{L}$ is *true* in x, written $(x, A)\mathbb{T}$, iff $\forall\alpha\in a$, $(x, \alpha)T$.

(DF7) Let a, $b\in\mathscr{L}$; then $a\subseteq b$ iff $\alpha\in a$, $\beta\in b$ and $\alpha\prec\beta$ is a partial order relation on $\mathscr{L}$ with respect to which we will denote, if they exist, by $\bigcap$ and $\bigcup$ the g.l.b. and the l.u.b., respectively.

Let I_α be the relative certain question of $\alpha\in\mathscr{Q}$. We have $\mathscr{S}_T(I_\alpha)=\mathscr{D}_\alpha$ and $\mathscr{S}_F(I_\alpha)=\{\varnothing\}$, i.e., $\mathscr{S}_T(O_\alpha)=\{\varnothing\}$. Therefore, all relative absurd questions are JP-equivalent to each other and form a unique JP-proposition denoted by $\mathbb{O}$. On the other hand, two relative trivial questions I_α and I_β are JP-equivalent iff $\mathscr{D}_\alpha=\mathscr{D}_\beta$, i.e., iff $\alpha(gpt)\beta$.

Proposition 3.2. Let $\langle\mathscr{S}, \mathscr{Q}; I, \tilde{}, \triangle\rangle$ be an L-pq-s; then the poset $\langle\mathscr{L}, \subseteq\rangle$ is a complete meet-lattice in which we have that

$$\text{g.l.b.}\{a_j\}\equiv\bigcap a_j=[\triangle\alpha_j]_{jp}\qquad\text{for}\quad \alpha_j\in a_j$$

Therefore, the JP-proposition

$$\mathbb{O}=\bigcap_{a\in\mathscr{L}} a$$

exists and is the minimum element of $\mathscr{L}$.

Proof. From $\mathscr{D}_{\triangle\alpha_j}=\bigcap\mathscr{D}_{\alpha_j}$ we have that $\mathscr{D}_{\triangle\alpha_j}\subseteq\mathscr{D}_{\alpha_k}$ for all k; moreover, from $\triangle\alpha_j(x)=\bigcap\alpha_j(x)=x$ it follows that $\alpha_j(x)=x$, i.e., $\triangle\alpha_j\prec\alpha_k$ for all k. Let γ be a question such that $\gamma\prec\alpha_J$ for all j; then $x\in\mathscr{D}_\gamma$ and $\gamma(x)=x$ implies $x\in\mathscr{D}_{\alpha_j}$ and $\alpha_J(x)=x$ for all j, that is, $\bigcap\alpha_j(x)\equiv\triangle\alpha_j(x)=x$, i.e., $\gamma\prec\triangle\alpha_j$. ■

Theorem 3.3. In an L-pq-s the following are equivalent:

(i) $\langle\mathscr{L}, \subseteq\rangle$ is a complete lattice in which

$$\text{l.u.b.}\{a_j\}\equiv\bigcup a_j=\bigcap\{b\in\mathscr{L}\colon(\forall j), (a_j\subseteq b)\}$$

(ii) The JP-proposition

$$\mathbb{1}:=\bigcup_{a\in\mathscr{L}} a$$

exists in $\mathscr{L}$.

(iii) A question $\hat{I}$ exists, called the *certain* question, such that $\mathscr{D}_{\hat{I}}=\mathscr{S}$ and $\hat{I}(x)=x$ for all $x\in\mathscr{S}$.

The certain question $\hat{I}$ is the unique question of the maximum element $\mathbb{1}$ of the lattice $\mathscr{L}$.

Proof. Let (i) be true; then $\mathbb{1}=\bigcup_{a\in\mathscr{L}} a$ exists in $\mathscr{L}$.

If (ii) is true, then the relation $a\subseteq\mathbb{1}$ holds for every $a\in\mathscr{L}$. This means that for every $\alpha\in\mathscr{Q}$ and every $\hat{I}\in\mathbb{1}$

$$\alpha\prec\hat{I} \qquad \text{and} \qquad I_\alpha\prec\hat{I}$$

Now, for every $x\in\mathscr{S}$, let us choose $\alpha\in\mathscr{Q}$ such that $x\in\mathscr{D}_\alpha$ (AX3); therefore, since $x\in\mathscr{D}_{I_\alpha}$ and $I_\alpha(x)=x$, for every $\hat{I}\in\mathbb{1}$ we have $x\in\mathscr{D}_{\hat{I}}$ and $\hat{I}(x)=x$.

Let now (iii) be true; then the set

$$\{b\in\mathscr{L}: (\forall j), (a_j\subseteq b)\}$$

is not empty since $[\hat{I}]_{jp}$ belongs to it. Owing to Proposition 3.2, the l.u.b. of any family $\{a_j\}$ of JP-propositions is just $\bigcap \{b\in\mathscr{L}: (\forall j), (a_j\subseteq b)\}$. ■

In the case of two questions $\{\alpha, \beta\}\in\mathscr{Q}$, Definition 2.8 assures the existence of the question, denoted by $\alpha\triangle\beta$, such that $\mathscr{D}_{\alpha\triangle\beta}=\mathscr{D}_\alpha\cap\mathscr{D}_\beta$ and

$$(\alpha\triangle\beta)(x)=\alpha(x)\cap\beta(x)$$

This question could be a relative absurd question $\alpha\triangle\beta\in\mathbb{O}$. For instance, let us suppose that $\mathscr{D}_\alpha\cap\mathscr{D}_\beta=\{\varnothing\}$. Hence, α and β are not performable together. Now we consider the JP-propositions $a=[\alpha]_{jp}$ and $b=[\beta]_{jp}$. Given any question γ, we have $\mathscr{S}_T(O_\gamma)\subseteq\mathscr{S}_T(\alpha)$ and $\mathscr{S}_T(O_\gamma)\subseteq\mathscr{S}_T(\beta)$, i.e., for every $\gamma\in\mathscr{Q}$, $O_\gamma\prec\alpha$ and $O_\gamma\prec\beta$. From this it follows that

$$\mathbb{O}=[O_\gamma]_{jp}\subseteq a, b$$

On the other hand, let δ be a question such that $\delta\prec\alpha$ and $\delta\prec\beta$. Let $x\in\mathscr{D}_\delta$ be such that $\delta(x)=x$; this implies, in particular, $x\in\mathscr{D}_\alpha$ and $x\in\mathscr{D}_\beta$, i.e., $x\in\mathscr{D}_\alpha\cap\mathscr{D}_\beta=\{\varnothing\}$. Then

$$\delta\prec\alpha, \quad \delta\prec\beta, \quad \text{and} \quad \delta(x) \quad \text{imply } x=\varnothing$$

Therefore, the following proposition holds.

Proposition 3.4. If two questions α and β are not performable together, then

$$[\alpha]_{jp}\cap[\beta]_{jp}=\mathbb{O}$$

Remark 3.5. In agreement with Piron (1976*b*), we have called JP-propositions the equivalence classes of questions; in the same way as Piron, we find that under the assumption of the existence of the certain question $\hat{I}\in\mathscr{Q}$ the set $\mathscr{L}$ of all JP-propositions turns out to be a complete lattice. However, it must be stressed that there is a profound difference between the physical interpretations of the meet $a \cap b$ of two JP-propositions, as defined in the present work, and Piron's one $a \cap_P b$. Indeed we have the following two physical interpretations of the formulas $(x, a \cap b)\mathbb{T}$ and $(x, a \cap_P b)\mathbb{T}$:

$(x, a \cap b)\mathbb{T}$ "On each physical sample $i \in x$, two questions $\alpha \in a$ and $\beta \in b$ *are performed together*, and the outcome yes is obtained for both α and β."

$(x, a \cap_P b)\mathbb{T}$ "If we perform (*not together*) one of two questions $\alpha \in a$ and $\beta \in b$ on each sample $i \in x$, the answer yes is obtained with certainty."

In the remaining part of the present section we shall assume that the involved L-pq-s satisfies one, and therefore all, of the statements of Theorem 3.3.

We recall that the approach we have outlined here is based on the **primitive notion** of an *individual sample* of a physical entity, from which the **derived notion** of the *preparation* of samples (as a subset of individual samples identified with the derived notion of *ensemble of samples*) has been formalized by the structure $\mathcal{S}$ of *selection procedures* based on a set M. We must underline that the assimilation of *states* of the Piron approach with *preparation procedures* is a real misunderstanding. Indeed, in Piron's approach, states are suitable equivalence classes of propositions [and so in our formalized context it turns out to be a new derived notion based on (DF4)].

We quote the definition of a *state* according to the JP way of thinking:

(DS) The *state* associated to a preparation procedure x is the set $\sigma(x)$ of all propositions (or properties) actually true (or certain) for the system prepared in x.

This definition matches with the following statement of Piron (1981): "If one given system has been prepared [x] in such a way that we can affirm that in the event of the experiment [$\alpha \in a$] the expected result would be certain [$(x, \alpha)T$, i.e., $(x, a)\mathbb{T}$], we will say that the corresponding property [associated to a] is an actual property of the system [in x], in opposition to the other properties which [in x] are only potential." In this case any individual sample prepared in x actually possesses all the properties corresponding to the propositions of the state $\sigma(x)$. We intend to present now the formal definition of the JP "state" and the related properties:

(DF8) The JP state associated to the preparation procedure x is the subset of propositions $\sigma(x) = \{a \in \mathscr{L} : (x, a)\mathbb{T}\}$.

(DF9) The set of all states is $\Sigma := \{\sigma(x) : x \in S\}$. The elements of Σ will be denoted by $u, v, w, \ldots$ (with indices, if necessary).

(DF10) A JP pure state is any JP state $\sigma(x)$ which is maximal in Σ.

(DF11) The set of all JP pure states is denoted by Σ_p.

(DF12) A pure preparation is any preparation $x_p \in S$ whose associated JP state $\sigma(x_p)$ is pure.

(DF13) We denote by S_p the set of all pure preparations; evidently, $\Sigma_p = \{\sigma(x_p) : x_p \in S_p\} = \sigma(S_p)$.

For any preparation $x \in \mathcal{S}$ the set $\sigma(x)$ is the collection of all JP propositions which are *actually true*, i.e., *true with certainty*, in x. This set determines all the physical properties that can be attributed with certainty to the samples of the physical entity prepared according to x, whether the state has been measured. Thus $\sigma(x)$ embodies the *amount of information* actually available for *any single sample* of the physical entity prepared according to x. Hence, if $x_p \in S_p$ is a pure preparation, the information available on any individual sample prepared according to x_p is maximal, i.e., $\sigma(x_p)$ embodies a *maximal amount of information.*

The following theorem collects some basic properties of JP states (Jauch and Piron, 1969; Cattaneo *et al.*, 1989; Cattaneo and Nisticò, 1992).

Theorem 3.6. The set Σ of all JP states satisfies the following properties:

(S1) If $a \in \sigma(x)$ and $a \subseteq b$, then $b \in \sigma(x)$.

(S2) If $\{a_i : i \in I\} \subseteq \sigma(x)$, then $\bigcap_{i \in I} a_i \in \sigma(x)$.

(S3) $\mathbb{O} \notin \sigma(x)$, $\mathbb{1} \in \sigma(x)$ for every $\sigma(x)$.

(S4) For any $a \in \mathcal{L}$, $a \neq \mathbb{O}$, there exists at least one $\sigma(x)$ such that $a \in \sigma(x)$.

If for every $x \in S$, we put

$$e(x) = \bigcap_{a \in \sigma(x)} a$$

[which, owing to (S2), exists], then $e(x) \in \sigma(x)$ and

$$\sigma(x) = \{a \in \mathcal{L} : e(x) \subseteq a\}$$

Hence, we can conclude that $\sigma(x)$ is characterized by $e(x)$. Furthermore, let us denote by $\mathcal{L}_a$ the set of all atoms in the lattice $\mathcal{L}$; whenever an atom $e \in \sigma(x) \cap \mathcal{L}_a$ exists, then $e = e(x)$.

Theorem 3.7. For every pure preparation $x_p \in S_p$, the proposition $e(x_p)$ characterizing the state $\sigma(x_p)$ is an atom of the lattice $\langle \mathcal{L}, \mathbb{O}, \subseteq \rangle$.

Conversely, for every atom e of the lattice $\langle \mathcal{L}, \mathbb{O}, \subseteq \rangle$ a pure preparation $x_p \in S_p$ exists such that $e(x_p) = e$.

It follows from Theorem 3.7 that every atom of $\mathscr{L}$ can be bijectively associated to a JP pure state according to the following one-to-one and onto correspondence:

$$\Sigma_p \equiv \mathscr{L}_a$$

$$\sigma(x_p) \leftrightarrow e(x_p)$$

The existence of this one-to-one mapping allows us to identify any pure state with the corresponding atom. While all the above definitions and results about JP states can be found in the work of JP (see, for instance, Jauch and Piron, 1969), the lack of any formalization of the notion of preparation neglects some important aspects which we now complete. Let $x, y \in \mathscr{S}$.

(DF14) The CGN equivalence relation on preparations is defined as $x \approx y$ iff $\sigma(x) = \sigma(y)$.
(DF15) A CGN state is any equivalence class $[x]_\approx := \{y : y \approx x\} = \{y : \sigma(y) = \sigma(x)\}$.
(DF16) The set of all CGN states is denoted by $\mathbf{S} := \{[x]_\approx : x \in \mathscr{S}\}$.
(DF17) The JP state associated to a CGN state is $\sigma([x]_\approx) := \sigma(y)$, whatever be $y \in [x]_\approx$.

Any CGN equivalence class of preparations $[x]_\approx = \{y : \sigma(y) = \sigma(x)\}$ is identifiable with the unique JP state $\sigma(x)$, which is defined as the JP state $\sigma([x]_\approx)$ of the whole equivalence class of preparations $[x]_\approx$. In symbols,

$$\Sigma \equiv \mathbf{S}$$

$$\sigma(x) \leftrightarrow [x]_\approx$$

Thus, for a JP state we can mean both the equivalence class of preparations $[x]_\approx$ and the set $\sigma([x]_\approx)$ of all propositions true (properties actual) in this state.

4. ORTHODOX LUDWIG EFFECTS AND CORRESPONDING PIRON QUESTIONS

In this section we investigate the connections between an L-pq-s and the orthodox Ludwig axiomatic foundation of quantum physics. The Ludwig approach to preparing–recording structures can be synthesized by the following structure based on M.

Definition 4.1. A complete Ludwig's preparing–recording structure (L-pr-s) on M is a triple $\mathbf{PR} := \langle \mathscr{U}, (\mathscr{R}_0, \mathscr{R}) \rangle$, where:

(PR1) $\mathscr{U}$ is a complete selection structure and $\mathscr{R}_0$, $\mathscr{R}$ are two selection structures on M.
(PR2) $\mathscr{R}_0 \subseteq \mathscr{R}$.

$\mathscr{U}$ is the set of *preparing procedures*, $\mathscr{R}_0$ the set of *recording methods*, and $\mathscr{R}$ the set of *recording procedures*. Following Ludwig, let us introduce the set of all *effect procedures*

$$\mathscr{F} := \{(b_0, b): b_0 \in \mathscr{R}_0,\ b \in \mathscr{R},\ b \subseteq b_0\}$$

Any effect procedure $f = (b_0, b) \in \mathscr{F}$ consists of the "recording procedure" $b \in \mathscr{R}$ and of the "recording counter" (or "recording apparatus") $b_0 \in \mathscr{R}_0$.

Remark 4.2. The set $\mathscr{F}$ contains the element $(\emptyset, \emptyset)$, called the *null effect*; we denote by $\mathscr{F}' := \mathscr{F} \backslash \{(\emptyset, \emptyset)\}$.

We quote Ludwig (1977):

> The recording process $[(b_0, b) \in \mathscr{R}_0 \times \mathscr{R},\ b \subseteq b_0]$ is characterized by two steps:
> (1) Construction and employment of the recording apparatus $[b_0 \in \mathscr{R}_0]$.
> (2) Selection according to signals which "appeared" $[b_+ = b \in \mathscr{R}]$ or "did not appear" $[b_- = b_0 \backslash b \in \mathscr{R}]$ on the recording apparatus employed.
> (...) Generally $\mathscr{R}$ is the set of all those selection procedures which are finer than the procedures of $\mathscr{R}_0$; finer by virtue of the influence of the microsystems on the apparatus, represented by the elements of $\mathscr{R}_0$.

Remark 4.3. It is straightforward to show that for any family $\{\mathscr{S}_j\}$ of selection structures (respectively, complete selection structures) based on the same set M, their set-theoretic intersection $\bigcap_j \mathscr{S}_j$ is a selection structure (respectively, complete selection structure) based on M.

Let Θ be any family of subsets of M; then $\mathscr{S}(\Theta) := \bigcap_j \mathscr{S}_j(\Theta)$, where $\{\mathscr{S}_j(\Theta)\}$ is the collection of all the selection structures (respectively, complete selection structures) containing Θ, is the selection structure *generated* by Θ.

Once given the above L-pr-s, we define the set

$$\Theta := \{a \cap b: a \in \mathscr{U},\ b \in \mathscr{R}\}$$

$\mathscr{S}(\Theta)$ is the complete selection structure generated by Θ. Let $a \in \mathscr{U}$ be a preparing procedure and $(b_0, b) \in \mathscr{F}$ an effect procedure; then we have the following *physical interpretation* of the formula "$x = a \cap b \in \Theta$ and $i \in x$":

> "An element $x = a \cap b \in \Theta$ is the set of all systems i prepared by the procedure $a \in \mathscr{U}$ *and* recorded by the procedure $b \in \mathscr{R}$" (Ludwig, 1977).

Proposition 4.4. For any $f = (b_0, b) \in \mathscr{F}$, setting

$$\mathscr{D}_f = \{a \cap b_0: a \in \mathscr{U}\} \subseteq \mathscr{S}(\Theta)$$

we have that:

(i1) $\varnothing \in \mathscr{D}_f$.
(i2) $\{(a_j \cap b_0) : j \in J\} \subseteq \mathscr{D}_f$ implies $\bigcap_j (a_j \cap b_0) \in \mathscr{D}_f$.

The mapping

$$\alpha_f: \quad \mathscr{D}_f \mapsto \mathscr{S}(\Theta), \quad a \cap b_0 \to \alpha_f(a \cap b_0) := a \cap b$$

is a question based on $\mathscr{S}(\Theta)$, called the L-question induced by the effect procedure f; the set of all such L-questions is denoted by $\mathscr{Q}(\mathscr{F})$.

In particular, for any given effect procedure $f = (b_0, b) \in \mathscr{F}$, there exist the two effect procedures $i_f = (b_0, b_0)$ and $o_f = (b_0, \varnothing)$ such that $\mathscr{D}_f = \mathscr{D}_{i_f} = \mathscr{D}_{o_f}$ and, for arbitrary $(a \cap b_0) \in \mathscr{D}_f$.

$$\alpha_{i_f}(a \cap b_0) = a \cap b_0 \qquad \text{and} \qquad \alpha_{o_f}(a \cap b_0) = \varnothing \tag{4.1}$$

(all α_{0_f} are null questions).

Let $f = (b_0, b)$ be an effect procedure; then $f^{\sim} := (b_0, b_0 \backslash b)$ is an effect procedure such that $\mathscr{D}_f = \mathscr{D}_{f^{\sim}}$ and, for arbitrary $(a \cap b_0) \in \mathscr{D}_f$,

$$\alpha_{f^{\sim}}(a \cap b_0) = a \cap b_0 \backslash a \cap b \tag{4.2}$$

Proof. $\varnothing \in \mathscr{U}$ implies $\varnothing = \varnothing \cap b_0 \in \mathscr{D}_f$. Moreover, $\bigcap (a_j \cap b_0) = (\bigcap a_j) \cap b_0$ proves (i2); i.e., $\mathscr{D}_f$ satisfies condition (i) in Definition 2.2. From $b \subseteq b_0$ it follows that $\alpha_f(a \cap b_0) \subseteq a \cap b_0$, which is (ii) of Definition 2.2. Lastly, if $a_1 \cap b_0$, $a_2 \cap b_0 \in \Theta$, then we have

$$\alpha_f[(a_1 \cap b_0) \cap (a_2 \cap b_0)] = b \cap (a_1 \cap a_2 \cap b_0) = (a_1 \cap b) \cap (a_2 \cap b)$$
$$= \alpha_f(a_1 \cap b_0) \cap (a_2 \cap b_0)$$

which proves (iii) of Definition 2.2. The other results are straightforward once one notes that for every $b_0 \in \mathscr{R}_0$, $(b_0, \varnothing) \in \mathscr{F}$ and $(b_0, b_0) \in \mathscr{F}$. ∎

Remark 4.5. In this way, to any effect procedure $f = (b_0, b)$ of Ludwig's approach we have associated a question

$$\alpha_f: \mathscr{D}_f \mapsto \mathscr{S}(\Theta), \; a \cap b_0 \to \alpha_f(a \cap b_0) := a \cap b$$

of Piron's approach.

Consistently with the nomenclature about questions (Section 2.1), the pair $(a \cap b_0, \alpha_f)$ is a *yes–no experiment* consisting of the *preparing process* $a \cap b_0$ and the question α_f, with associated *recording process* $a \cap b$. To be precise, the preparing procedure a, which describes "the procedure by which physical systems are prepared," and the recording methods b_0, which "represents the construction and employment of a recording apparatus" (Ludwig, 1977), give rise to the *preparing process* $a \cap b_0$; the test of the question α_f on each individual sample prepared in this process furnishes the *measuring* *process* $a \cap b$ consisting of all the samples which have produced the answer "yes."

The mapping $f \to \alpha_f$, in general, is neither one-to-one nor onto; it could happen that different effect procedures give rise to a unique question and that there is some question for $\mathscr{S}(\Theta)$ which cannot be obtained by any effect procedures according to the now outlined construction. Because of the physical interpretation of (b_0, b) we consider as L-questions only those questions for $\mathscr{S}(\Theta)$ induced by effect procedures.

Theorem 4.6. Let $\mathbf{PR} := \langle \mathscr{U}, (\mathscr{R}_0, \mathscr{R}) \rangle$ be a complete Ludwig preparing–recording structure; then, for

(a) $I: \mathscr{Q}(\mathscr{F}) \mapsto \mathscr{Q}(\mathscr{F})$ is the unary operator associating to any L-question α_f the L-question $I_{\alpha_f} := \alpha_{i_f}$ of (4.1)

(b) $^\sim: \mathscr{Q}(\mathscr{F}) \mapsto \mathscr{Q}(\mathscr{F})$ is the unary operator associating to any L-question α_f the L-question $(\alpha_f)^\sim := \alpha_{f^\sim}$ of (4.2)

$\mathbf{PQ(PR)} := \langle \mathscr{S}(\Theta), \mathscr{Q}(\mathscr{F}); \mathbf{O}; I, {}^\sim \rangle$ is a preparing–question structure according to Definition 2.4 such that for any family of effect procedures $\{f_j = (b_0^{(j)}, b^{(j)})\}$, the effect procedure $\triangle f_j := (\bigcap_j b_0^{(j)}, \bigcap_j b^{(j)})$ exists which satisfies

$$\alpha_{\triangle f_j}(a \cap (\bigcap_j b_0^{(j)})) = \bigcap_j \alpha_{f_j}(a \cap b_0^{(j)}) \tag{4.3}$$

In this way we have that **PQ(PR)** is an L preparing–question structure according to Definition 2.11.

Proof. From $b \in \mathscr{R}$ and $b_0 \in \mathscr{R}_0 \subseteq \mathscr{R}$ it follows that $b_0 \backslash b \in \mathscr{R}$ and so $(b_0, b \backslash b_0)$ is an effect procedure. Moreover, from $\alpha_f(a \cap b_0) = a \cap b$ we get that

$$(\alpha_f)^\sim(a \cap b_0) = a \cap b_0 \backslash a \cap b = a \cap (b_0 \backslash b)$$

and so

$$\alpha_{f^\sim}(a \cap b_0) = a \cap (b_0 \backslash b) = (\alpha_f)^\sim(a \cap b_0)$$

Moreover,

$$\begin{aligned} \alpha_{\triangle f_j}(a \cap (\bigcap_j b_0^{(j)})) &= a \cap (\bigcap_j b^{(j)}) = \bigcap_j (a \cap b^{(j)}) \\ &= \bigcap_j \alpha_{f_j}(a \cap b_0^{(j)}) \quad \blacksquare \end{aligned}$$

Remark 4.7. Let $\forall j$, $b_0^{(j)} = b_0$, i.e., $\{f_j = (b_0, b^{(j)})\}$; then $\bigcap_j b_0^{(j)} = b_0$ and (4.3) assumes the form

$$\alpha_{\triangle f_j}(a \cap b_0) = \bigcap_j \alpha_{f_j}(a \cap b_0)$$

Definition 4.8. An effect procedure $f = (b_0, b)$ is said to be *true* (respectively, *false*) in the preparing procedure a, written $(a, (b_0, b))T$ [respectively, $(a, (b_0, b))F$], iff $(a \cap b_0, \alpha_f)T$ [respectively, $(a \cap b_0, \alpha_f)F$]. Hence,

$$(a, (b_0, b))T \quad \text{iff} \quad a \cap b = a \cap b_0$$

$$(a, (b_0, b))F \quad \text{iff} \quad a \cap b = \varnothing$$

Remark 4.9. We can prove that

$$(a, f)T \quad \text{and} \quad a_1 \subseteq a \quad \text{imply} \quad (a_1, f)T$$

Indeed, $a_1 \subseteq a$ and $a \cap b_0 = a \cap b$ imply

$$\begin{aligned} a_1 \cap b_0 &= (a_1 \cap a) \cap b_0 = a_1 \cap (a \cap b_0) \\ &= a_1 \cap (a \cap b) = (a_1 \cap a) \cap b \\ &= a_1 \cap b \end{aligned}$$

Definition 4.10. Two effect procedures $f' = (b_0', b')$ and $f'' = (b_0'', b'')$ are said to be *performable together*, written $f'(pt)f''$, iff the corresponding questions $\alpha_{f'}$ and $\alpha_{f''}$ are performable together. Thus

$$f'(pt)f'' \quad \text{iff} \quad \exists a', a'' \in \mathscr{U}, \quad a' \cap b_0' = a' \cap b_0''$$

Lastly, we characterize a particular class of effect procedures.

Proposition 4.11. Let $f = (b_0, b)$ be an effect procedure; then $\forall a \in \mathscr{U}$, $a \cap b \in \mathscr{U}$ implies that the induced question ϕ_f is a *filter*, i.e., it satisfies $\phi_f(\mathscr{D}_f) \subseteq \mathscr{D}_f$; in turn this condition is equivalent to the fact that ϕ is an idempotent filter.

Proof. Let $a \cap b \in \mathscr{U}$ for every $a \in \mathscr{U}$; then, for arbitrary $a \in \mathscr{U}$, $\phi_f(a \cap b_0) = a \cap b \ (\in \mathscr{U}) = (a \cap b) \cap b_0 \in \mathscr{D}_f$. On the other hand, if ϕ_f is a filter, then for every $a \in \mathscr{U}$ we have that $a \cap b = \phi_f(a \cap b_0) \in \mathscr{D}_f$; from this we get

$$\phi_f^2(a \cap b_0) = \phi_f(a \cap b) = \phi_f((a \cap b) \cap b_0) = (a \cap b) \cap b = a \cap b = \phi_f(a \cap b_0)$$

Conversely, if ϕ_f is idempotent, then for every $a \cap b_0 \in \mathscr{D}_f$, $\phi_f(a \cap b_0) = \phi_f^2(a \cap b_0) = \phi_f(a \cap b)$ [and so necessarily it must be that $a \cap b \in \mathscr{D}_f$, i.e., there exists $a^* \in \mathscr{U}$ such that $a \cap b = a^* \cap b_0$], from which we get $\phi_f(a \cap b_0) = \phi_f(a \cap b) = (a \cap b) \cap b_0 = a \cap b \in \mathscr{D}_f$, obtaining that $\phi_f(a \cap b_0) \in \mathscr{D}_f$. ■

In conclusion, in this section we have recovered a Piron preparing–question structure from a complete Ludwig preparing–recording structure.

5. STATISTICAL SELECTION–QUESTION STRUCTURE

In the theory developed up to now we have considered Ludwig's selection structure without any statistical notion. In this section we investigate some preliminary results involving probability notions.

Definition 5.1. A structure of pre-statistical selection procedures or briefly a *pre-statistical selection structure* based on the set M is any pair $\langle \mathscr{S}, \nu \rangle$ where $\mathscr{S}$ is a selection structure for M; and $\nu: \mathscr{S} \mapsto \mathbb{R}_+$, called an *absolute quantum counter*, is a mapping satisfying the following conditions:

(AM1) For any x_1, $x_2 \in \mathscr{S}$, $x_1 \cap x_2 = \varnothing$ and $x_1 \cup x_2 \in \mathscr{S}$ imply $\nu(x_1 \cup x_2) = \nu(x_1) + \nu(x_2)$.

(AM2) $\nu(x) = 0$ iff $x = \varnothing$.

Remark 5.2. Neumann (1983) has called an *effective measure* any mapping $\nu: \mathscr{S} \mapsto \mathbb{R}_+$ which satisfies (AM1) and (AM2). Any effective measure satisfies the further property:

(AM3) Let $x, y \in \mathscr{S}$; then $x \subseteq y$ implies $\nu(x) \leq \nu(y)$.

Indeed, from $x \subseteq y$ we get $y = x \cup (y \backslash x)$ and so (AM3) follows from (AM1).

The *physical interpretation* of the formula $\nu(x)$ is as follows:

The real quantity $\nu(x)$ is the total number of individual samples [or the intensity (Mielnik, 1969)] of the ensemble prepared in x.

An effective measure ν on $\mathscr{S}$ induces a *relative probability* (partially defined) in $\mathscr{S} \times \mathscr{S}$ according to the following results (Neumann, 1983).

Proposition 5.3. Let $\mathscr{I}(\mathscr{S}) = \{(x, y) : x, y \in \mathscr{S}, y \subseteq x, \text{ and } x \neq \varnothing\}$; then the mapping

$$\lambda: \quad \mathscr{I}(\mathscr{S}) \mapsto [0, 1]$$

defined by

$$\lambda(x, y) := \frac{\nu(y)}{\nu(x)}$$

satisfies the following conditions:

1. $x_1, x_2 \in \mathscr{S}$, $x_1 \cap x_2 = \varnothing$, $x_1 \cup x_2 \in \mathscr{S}$ imply

$$\lambda(x_1 \cup x_2, x_1) + \lambda(x_1 \cup x_2, x_2) = 1$$

2. $x_1, x_2, x_3 \in \mathscr{S}$, $x_3 \subseteq x_2 \subseteq x_1$, $x_2 \neq \varnothing$ imply

$$\lambda(x_1, x_3) = \lambda(x_2, x_3)\lambda(x_1, x_2)$$

3. $x_1, x_2 \in \mathscr{S}$, $x_2 \subseteq x_1$, $x_2 \neq \varnothing$ imply $\lambda(x_1, x_2) \neq 0$.
4. For every $x \in \mathscr{S}$

$$\lambda(x, x) = 1, \qquad \lambda(x, \varnothing) = 0$$

5. $x_1, x_2, x_3 \in \mathscr{S}$, $x_2 \subseteq x_1$, $x_3 \subseteq x_1$, $x_2 \cap x_3 = \varnothing$ imply

$$\lambda(x_1, x_2 \cup x_3) = \lambda(x_1, x_2) + \lambda(x_1, x_3)$$

Proof. We prove point 5 only; the proofs of the other points are straightforward. Since $x_1 \cup x_3 = x_1 \backslash (x_1 \backslash x_2 \cap x_1 \backslash x_3)$, from Definition 5.1 it follows that $x_2 \cup x_3 \in \mathscr{S}$. Moreover, from x_2, $x_3 \subseteq x_1$ we have that $x_2 \cup x_3 \subseteq x_1$. From these results we get

$$\lambda(x_1, x_2 \cup x_3) = \frac{\nu(x_2 \cup x_3)}{\nu(x_1)} = \frac{\nu(x_2) + \nu(x_3)}{\nu(x_1)}$$

$$= \frac{\nu(x_2)}{\nu(x_1)} + \frac{\nu(x_3)}{\nu(x_1)} \qquad \blacksquare$$

As a consequence of points 1–3 of Proposition 5.3 we have that the pair $\langle \mathscr{S}, \lambda \rangle$ is a *structure of statistical selection procedure* (or briefly, a *statistical selection structure*) of Ludwig's approach: "$\lambda(x, y)$ is usually called the *probability* of y relative to x. $\lambda(x, y)$ is the mathematical picture of the frequency with which y selects relative to x (. . .). With this 'interpretation' of $\lambda(x, y)$ at hand, the reader may easily check the 'physical' significance of [1–3, Proposition 5.3] (. . .). From [1–3] we obtain [4 and 5]" (Ludwig, 1977). Note that in any "laboratory" $\langle x, \mathscr{S}(x) \rangle$, where $x \in \mathscr{S}$ and $\mathscr{S}(x) := \{y : y \in \mathscr{S} \text{ and } y \subseteq x\}$ (which is a Boolean algebra with unit for x), the mapping $\tilde{\mu} : \mathscr{S}(x) \mapsto [0, 1]$ defined by the law $\forall y \in \mathscr{S}(x)$, $\tilde{\mu}(y) := \lambda(x, y)$ is a finitely additive (positive) measure of mass one [which can be considered a probability measure on the Boolean algebra $\mathscr{S}(x)$].

Definition 5.4. A *statistical preparation–question structure* (briefly, s-pq-s) is any triple $\langle (\mathscr{S}, \nu), \mathscr{Q} \rangle$ where $\langle \mathscr{S}, \nu \rangle$ is a statistical selection structure; and $\langle \mathscr{S}, \mathscr{Q} \rangle$ is a preparation–question structure.

Let $\langle (\mathscr{S}, \nu), \mathscr{Q} \rangle$ be an s-pq-s. Then we introduce the partial mapping

$$p : (\mathscr{S} \times \mathscr{Q})_p \mapsto [0, 1]$$

defined for every $\alpha \in \mathscr{Q}$ and every $x \in \mathscr{D}'_\alpha$ by

$$p(x, \alpha) := \lambda(x, \alpha(x)) = \frac{\nu(\alpha(x))}{\nu(x)}$$

Putting

$$\mathscr{S}(\mathscr{Q}) = \bigcap_{\alpha \in \mathscr{Q}} (\mathscr{D}'_\alpha)$$

we obtain that the restriction of p to $\mathscr{S}(\mathscr{Q}) \times \mathscr{Q}$ is total.

Given a selection procedure $x \in \mathscr{D}'_\alpha$, the number $p(x, \alpha)$ is physically interpreted as the probability that the physical entity prepared in x gives the answer yes to the question α. We have the following result.

Proposition 5.5. Let $\langle (\mathscr{S}, \nu), \mathscr{Q} \rangle$ be an s-pq-s; then for every $\alpha \in \mathscr{Q}$ and every $x \in \mathscr{D}'_\alpha$

$$p(x, \alpha) = 1 \quad \text{iff} \quad (x, \alpha)T \tag{5.1}$$

Proof. Suppose that $p(x, \alpha) = 1$, i.e., $\nu(\alpha(x)) = \nu(x)$. We have

$$x = \alpha(x) \cup x \backslash \alpha(x), \qquad \text{where} \qquad \alpha(x) \cap x \backslash \alpha(x) = \varnothing \tag{5.2}$$

From (AM1) it follows that

$$\nu(x) = \nu(\alpha(x)) + \nu(x \backslash \alpha(x)) = \nu(x) + \nu(x \backslash \alpha(x))$$

Therefore, $\nu(x \backslash \alpha(x)) = 0$ and so, by (AM2), we get $x \backslash \alpha(x) = \varnothing$; substituting this result in (5.2), we obtain $\alpha(x) = x$. The converse is trivial. ■

The result (5.1) says that in the context of an s-pq-s our formalization of "true with certainty" [$(x, \alpha)T$ iff $\alpha(x) = x$] is equivalent to "probability one" [$p(x, \alpha) = 1$], as for most quantum physicists and mathematicians.

In the context of an L-pr-s introduced in Section 4, $\mathbf{PR} = \langle \mathscr{U}, (\mathscr{R}_0, \mathscr{R}) \rangle$, if the complete selection structure $\mathscr{S}(\Theta)$ generated by

$$\Theta = \{a \cap b : a \in \mathscr{U}, b \in \mathscr{R}\}$$

is a pre-statistical selection structure with absolute quantum counter $\nu_{\mathscr{S}(\Theta)}$, then according to Ludwig, we can introduce the mapping

$$\mu : \mathscr{U} \times \mathscr{F} \mapsto [0, 1]$$

defined for every preparing procedure $a \in \mathscr{U}$ and every effect $f = (b_0, b) \in \mathscr{F}$ by

$$\mu(a, (b_0, b)) := \lambda_{\mathscr{S}(\Theta)}(a \cap b_0, a \cap b) = \frac{\nu_{\mathscr{S}(\Theta)}(a \cap b)}{\nu_{\mathscr{S}(\Theta)}(a \cap b_0)} = p_{\mathscr{S}(\Theta)}(a \cap b_0, a_{(b_0, b)})$$

which is equal to the probability of occurrence of the question $\alpha_{(b_0,b)}$ in the preparation $a \cap b_0$ of the associated L-pq-s. Recall that $(a \cap b_0, \alpha_{(b_0,b)})T$ iff $a \cap b = a \cap b_0$ (see Definition 4.8).

6. CONCLUSION

In the present work we constructed a model of the basic Piron theory (without Axioms C, P, A) in terms of Ludwig's selection structure (without statistical notions); we also investigated some preliminary extensions to Ludwig's statistical selection structure. As a subject of further interest, one should show in which way a modelization of the complete Piron theory is related to Ludwig's approach; precisely, one should point out which modifications (if present) are needed in Ludwig's formalization so that Piron's Axioms C, P, A turn out to be theorems of the proposed model. We think that this study can lead to a deeper knowledge of the two approaches.

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